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A New Mixture Lomax Distribution and Its Application

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Abstract: In this paper, we propose a new mixture Lomax distribution for positive continuous random variable. The new mixture Lomax distribution has two component distributions which are Lomax distribution and length biased Lomax distribution. We have derived and studies in probability properties which include the probability density function, cumulative distribution function, survival function, hazard function, moment about origin, mean, variance, coefficient of skewness and coefficient of kurtosis. Next, we study the estimation parameter of new mixture Lomax distribution by using maximum likelihood estimation. Finally, application of new mixture Lomax distribution is illustrated by real data set which is analyzed using Akaike's information criterion (AIC). It is shown that the proposed distribution fits much better than some other existing Lomax distributions.

Keywords: Mixture Lomax distribution, Length biased Lomax distribution.

1. Introduction

Lomax distribution was introduced by Lomax (1954). Lomax distribution is Pareto distribution Type II which is heavy tailed distributions for positive continuous random variable. It used in business, economics, queuing theory and Internet traffic modeling (Chen et al., 2015). In the literature, some extension of Lomax distribution are available such as Gamma-Lomax distribution by Cordeiro el al. (2013), Transmuted exponentiated Lomax distribution by Ashour and Eltehiwy (2013), Poisson-Lomax distribution by Al-Zahrania and Sagorb (2014), Power Lomax distribution by El-Houssainy et al. (2016) and Length-biased weighted Lomax distribution by Afaq et al. (2016).

Mixture distributions are popular tools for generating flexible distribution with good statistical properties. Specifically, let $0 < p < 1$ and $f_{X_1}(x)$ and $f_{X_2}(x)$ are the probability density function of the random variables X_1 and X_2, respectively, then the probability density function of the X (being a mixture of X_1 and X_2) is expressed as

$$f_X(x) = p\, f_{X_1}(x) + (1-p) f_{X_2}(x), \quad x > 0.$$

In the literature, some mixture distribution of several distributions are available such as mixture inverse gaussian distribution (Jorgensen et al. ,1991 ; Balakrishnan et al., 2009), Lindley distribution (Ghitany et al. 2008), three parameter crack distribution (Bowonrattanaset, 2011) and two parameter crack distribution (Pornpop et al.,2014). The mixture distribution with weight parameter (p) has many desirable properties in some applications, parameter estimation may still have problems. Jorgensen et al. (1991) and Gupta and Akman (1995) have mentioned that solving nonlinear simultaneous equations are non-trivial issues. Then, Bowonrattanaset (2011) showed the estimates of p are out of the closed interval [0,1] .

Mixture Lomax distribution has two component distributions which are Lomax distribution and length biased Lomax distribution. The probability density function of Lomax distribution (Aryal and Tsokos, 2011) and Length-biased Lomax distribution (Subba Rao et al., 2014), respectively , are given by

$$f_{X_1}(x) = \frac{q}{b\left[1 + x/b\right]^{1+q}} \quad \text{and}$$

$$f_{X_2}(x) = \frac{q(q-1)x}{b^2\left[1 + x/b\right]^{1+q}}, \quad x > 0, b, q > 0 .$$

Mixture Lomax distribution has two component distributions which are Lomax distribution and length biased Lomax distribution with probabilities p and $1-p$, respectively, as follows:

$$f(x) = p\frac{q}{b\left[1 + x/b\right]^{1+q}} + (1-p)\frac{q(q-1)x}{b^2\left[1 + x/b\right]^{1+q}}$$

where $x > 0, 0 < p < 1, b, q > 0.$

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Therefore, in order to solve such problems, a new weight parameter is considered. We introduce a new mixture Lomax distribution which is obtained by adding a new weight parameter. In section 2, we present probability density function, cumulative distribution function, survival function, hazard function, moment about origin, mean, variance, coefficient of skewness and coefficient of kurtosis. In section 3, we estimate parameter of new mixture Lomax distribution by using maximum likelihood estimation method. In section 4, we present an illustrative example with estimated parameters. Finally, we conclude in section 5.

2. New mixture Lomax distribution

In this section, we develop the new mixture Lomax distribution. For this new distribution, we present probability density function, cumulative distribution function, survival function, hazard function, moment about origin are given , respectively, in Theorem 1 – 5.

2.1 Probability density function of new mixture Lomax distribution

Definition 1. Let X_1 and X_2 random variables are independent and identically distributed (i.i.d.). X_1 is a random variable of Lomax distribution and X_2 is a random variable of length biased Lomax distribution. The probability density function of new mixture Lomax distribution (X) by the mixture between X_1 and X_2 with parameter $b > 0$ and $q > 0$ is defined as

$$f_X(x;b,q) = \left(\frac{b}{1+b}\right) f_{X_1}(x;b,q) + \left(\frac{1}{1+b}\right) f_{X_2}(x;b,q).$$

Theorem 1 . Let X is a random variable of new mixture Lomax distribution with $b>0$ and $q>0$. The probability density function of new mixture Lomax distribution is

$$f(x) = \frac{q(b^2+qx-x)}{(b^3+b^2)(1+x/b)^{1+q}}$$

where $x>0$, $b>0$ and $q>0$.

Proof of Theorem 1. In Definition 1, The probability density function of new mixture Lomax distribution is given as

$$f_X(x;b,q) = \frac{b}{1+b}\cdot f_{X_1}(x;b,q) + \frac{1}{1+b}\cdot f_{X_2}(x;b,q)$$

$$f(x) = \frac{b}{1+b}\cdot\left(\frac{q}{b[1+(x/b)]^{1+q}}\right) + \frac{1}{1+b}\cdot\left(\frac{q(q-1)x}{b^2\left[1+x/b\right]^{1+q}}\right).$$

Therefore, the probability density function of new mixture Lomax distribution is given by the expression

$$f(x) = \frac{q(b^2+qx-x)}{(b^3+b^2)(1+x/b)^{1+q}}.$$

The shape of probability density new mixture Lomax function is shown in Figure 1.

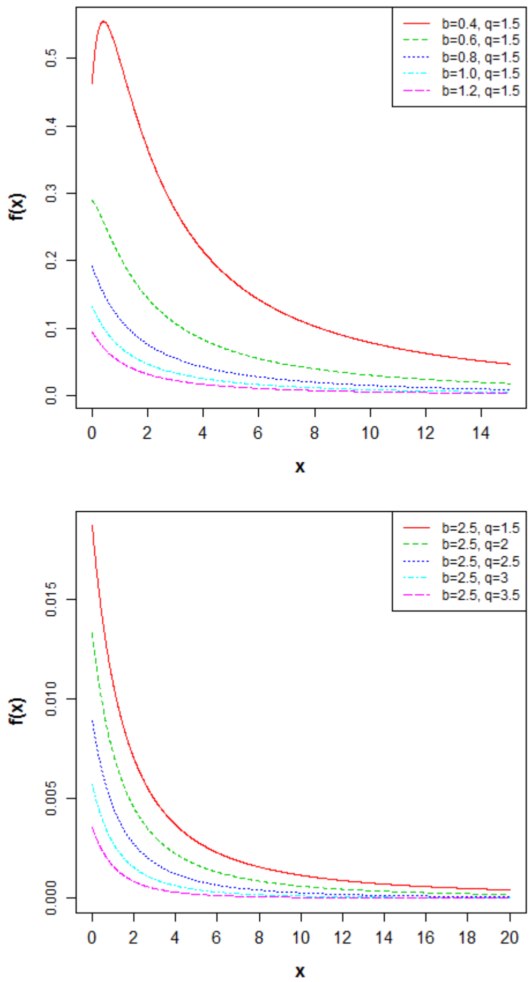

Figure 1 . Plot of the probability density new Lomax function: versus x for selected b and q .

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2.2 Cumulative distribution function of new mixture Lomax distribution

Theorem 2 . Let X is a random variable of new mixture Lomax distribution with $b > 0$ and $q > 0$. The cumulative distribution function of new mixture Lomax distribution is given as

$$F(x) = \frac{1}{1+b} \cdot \left(1 + b - \left(1 + b + \frac{qx}{b}\right)\left(1 + \frac{x}{b}\right)^{-q} \right)$$

where $x > 0$, $b > 0$ and $q > 0$.

Proof of Theorem 2. If X is a random variable of new mixture Lomax distribution, then cumulative distribution function of new mixture Lomax distribution can be given as

$$F(x) = \int_0^x f(t;b,q)\,dt$$

$$F(x) = \frac{b}{1+b}\int_0^x f_{X_1}(t;b,q)\,dt + \frac{1}{1+b}\int_0^x f_{X_2}(t;b,q)\,dt$$

$$F(x) = \frac{b}{1+b} \cdot F_{X_1}(x;b,q) + \frac{1}{1+b} \cdot F_{X_2}(x;b,q)$$

The cumulative distribution function of (Aryal and Tsokos, 2011) and length biased Lomax distribution (Subba Rao et al., 2014) , respectively, are as follows:

$$F(x_1) = 1 - \left(1 + \frac{x_1}{b}\right)^{-q}$$

and $F(x_2) = 1 - \left(1 + \frac{qx_2}{b}\right)\left(1 + \frac{x_2}{b}\right)^{-q}$.

The following result has obtained

$$F(x) = \frac{1}{1+b} \cdot \left(1 + b - \left(1 + b + \frac{qx}{b}\right)\left(1 + \frac{x}{b}\right)^{-q} \right).$$

The shape of cumulative distribution function of new mixture Lomax distribution is shown in Figure 2.

2.3 Survival function of new mixture Lomax distribution

Theorem 3 . Let X is a random variable of new mixture Lomax distribution with $b > 0$ and $q > 0$. The survival function of new mixture Lomax distribution is given as

$$S(x) = 1 - \frac{1}{1+b} \cdot \left(1 + b - \left(1 + b + \frac{qx}{b}\right)\left(1 + \frac{x}{b}\right)^{-q} \right).$$

Proof of Theorem 3. The cumulative distribution function of new mixture Lomax distribution is

$$F(x) = \frac{1}{1+b} \cdot \left(1 + b - \left(1 + b + \frac{qx}{b}\right)\left(1 + \frac{x}{b}\right)^{-q} \right).$$

The survival function is obtained by solving

$$S(x) = \int_x^\infty f(t)\,dt = 1 - F(x).$$

Then, survival function of new mixture Lomax distribution is

$$S(x) = 1 - \left[\frac{1}{1+b} \cdot \left(1 + b - \left(1 + b + \frac{qx}{b}\right)\left(1 + \frac{x}{b}\right)^{-q} \right) \right].$$

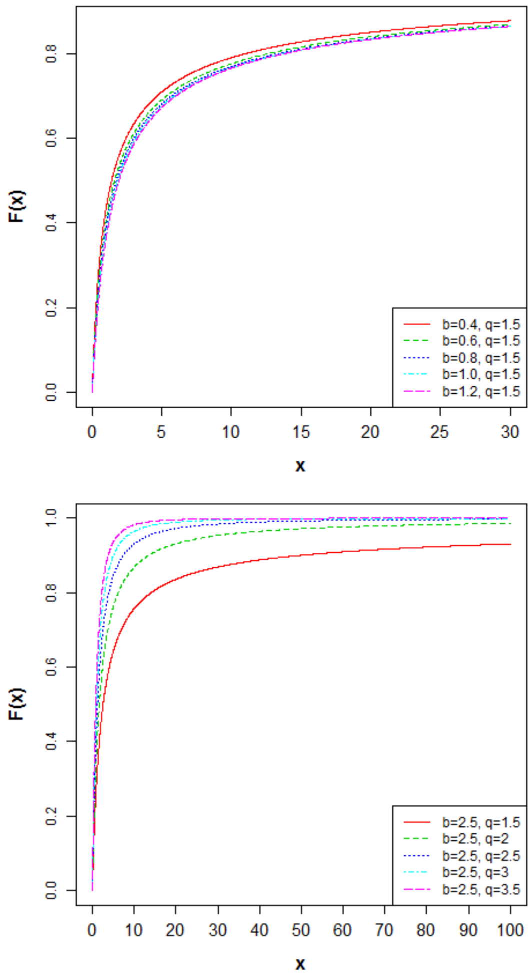

Figure 2 . Plot of cumulative distribution function of new mixture Lomax distribution plots function: versus x for selected b and q .

Next, the shape of survival new mixture Lomax function is shown in Figure 3.

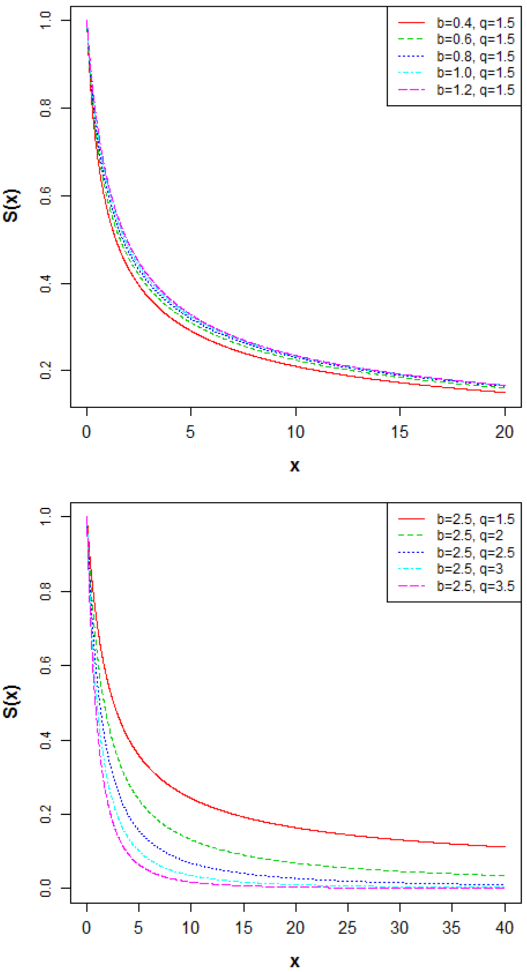

Figure 3 . Plot of the survival new mixture Lomax function: versus x for selected b and q .

2.4 Hazard function of new mixture Lomax distribution

Theorem 4 . Let X is a random variable of new mixture Lomax distribution with $b>0$ and $q>0$. The hazard function of new mixture Lomax distribution is given as

$$h(x)=\left[q(b^2+qx-x)\right]/\left[\left(b^3+b^2\right)(1+x/b)^{1+q}\right.$$
$$\left.\times\left(1-\frac{1}{1+b}\cdot\left(1+b-\left(1+b+\frac{qx}{b}\right)\left(1+\frac{x}{b}\right)^{-q}\right)\right)\right].$$

Proof of Theorem 4. The probability density function and survival function of new mixture Lomax distribution are

$$f(x)=\frac{q(b^2+qx-x)}{(b^3+b^2)(1+x/b)^{1+q}}$$ and

$$S(x)=1-\left[\frac{1}{1+b}\cdot\left(1+b-\left(1+b+\frac{qx}{b}\right)\left(1+\frac{x}{b}\right)^{-q}\right)\right]$$

.

The hazard function is obtained by solving

$$h(x)=\frac{f(x)}{S(x)}.$$

Then, we get

$$h(x)=\left[q(b^2+qx-x)\right]/\left[\left(b^3+b^2\right)(1+x/b)^{1+q}\right.$$
$$\left.\times\left(1-\frac{1}{1+b}\cdot\left(1+b-\left(1+b+\frac{qx}{b}\right)\left(1+\frac{x}{b}\right)^{-q}\right)\right)\right].$$

The shape of hazard new mixture Lomax function is shown in Figure 4.

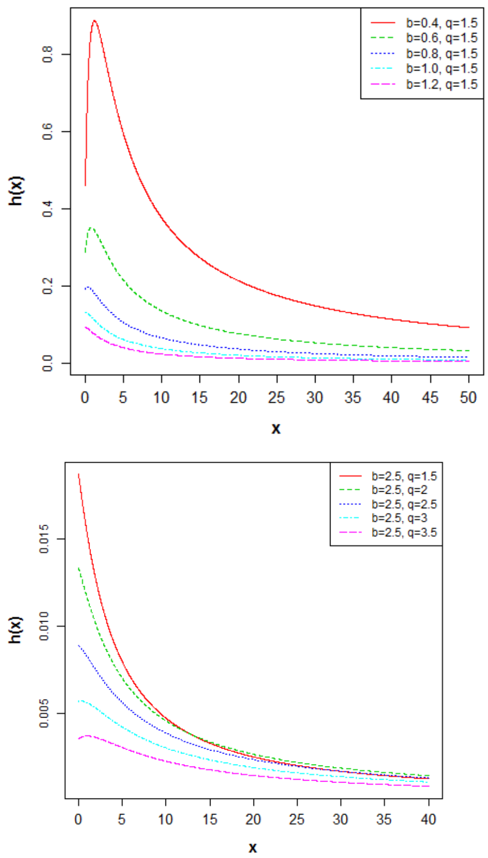

Figure 4. Plot of the hazard new mixture Lomax function: versus x for selected b and q .

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Theorem 5. The k th moment about the origin of a random variable X, where $X \sim NML(b,q)$ is given by the following:

$$E(X^k)=\frac{qb^{k+1}}{1+b}\sum_{r=0}^{k}\binom{k}{r}(-1)^{r+1}\left(\frac{1}{k-r-q}\right)+\frac{q(q-1)b^k}{1+b}\sum_{r=0}^{k+1}\binom{k+1}{r}(-1)^{r+1}\left(\frac{1}{k-r-q+1}\right).$$

Proof of Theorem 5. The k th moment about origin of X can be determined by direct integration using the probability density function, we have

$$E\left(X^k\right)=\int_0^\infty x^k f(x)dx=\int_0^\infty x^k \frac{q(b^2+qx-x)}{\left(b^3+b^2\right)\left(1+x/b\right)^{1+q}}dx$$

$$E\left(X^k\right)=\frac{q}{1+b}\int_0^\infty\left(\frac{x^k}{[1+(x/b)]^{1+q}}\right)dx+\frac{q(q-1)}{(1+b)}\int_0^\infty\left(\frac{x^{k+1}}{b^2\left[1+x/b\right]^{1+q}}\right)dx$$

$$=\frac{q}{1+b}\int_1^\infty b^k(t-1)^k t^{-(1+q)}b\ dt+\frac{q(q-1)}{(1+b)\ b^2}\int_1^\infty b^{k+1}(t-1)^{k+1}t^{-(1+q)}b\ \ dt$$

$$=\frac{qb^{k+1}}{1+b}\int_1^\infty (t-1)^k t^{-(1+q)}\ dt+\frac{q(q-1)b^k}{1+b}\int_1^\infty (t-1)^{k+1}t^{-(1+q)}\ dt$$

$$=\frac{qb^{k+1}}{1+b}\int_1^\infty\sum_{r=0}^{k}\binom{k}{r}t^{k-r}(-1)^r t^{-(1+q)}\ dt+\frac{q(q-1)b^k}{1+b}\int_1^\infty\sum_{r=0}^{k+1}\binom{k+1}{r}t^{k+1-r}(-1)^r t^{-(1+q)}\ dt$$

$$=\frac{qb^{k+1}}{1+b}\sum_{r=0}^{k}\binom{k}{r}(-1)^r\int_1^\infty t^{k-r-q-1}\ dt+\frac{q(q-1)b^k}{1+b}\sum_{r=0}^{k+1}\binom{k+1}{r}(-1)^r\int_1^\infty t^{k-r-q}\ dt$$

$$E(X^k)=\frac{qb^{k+1}}{1+b}\sum_{r=0}^{k}\binom{k}{r}(-1)^{r+1}\left(\frac{1}{k-r-q}\right)+\frac{q(q-1)b^k}{1+b}\sum_{r=0}^{k+1}\binom{k+1}{r}(-1)^{r+1}\left(\frac{1}{k-r-q+1}\right)$$

Corollary 1. Let X is random variable of new mixture Lomax distribution. Then the first - four moments of X are given, respectively, as follows:

$$E(X)=\frac{b^2}{(1+b)(q-1)}+\frac{2b}{(1+b)(q-2)},where\ q>2$$

$$E(X^2)=\frac{2b^3}{(1+b)(q-1)(q-2)}+\frac{6b^2}{(1+b)(q-2)(q-3)}$$

where $q>3$

$$E(X^3)=\frac{6b^4}{(1+b)(q-1)(q-2)(q-3)}+\frac{24b^3}{(1+b)(q-2)(q-3)(q-4)},where\ q>4$$

$$E(X^4)=\frac{24b^5}{(1+b)(q-1)(q-2)(q-3)(q-4)}+\frac{120b^4}{(1+b)(q-2)(q-3)(q-4)(q-5)},where\ q>5.$$

Proof of Corollary 1. Applying k th moment about the origin of random variable of new mixture Lomax distribution for $k=1,2,3$ and $k=4$ yields the desired results.

Based on the results given in relations of k th moment about the origin of random variable of new mixture Lomax distribution for $k=1,2,3$ and $k=4$, the mean, variance, coefficient of skewness and coefficient of kurtosis are given , respectively, as follows:

1) Mean

$$E(X)=\frac{b^2}{(1+b)(q-1)}+\frac{2b}{(1+b)(q-2)}$$

where $q>2$

2) Variance

$$Var(X)\ =\ E\left(X^2\right)-\left[E(X)\right]^2$$

$$=\frac{2b^3}{(1+b)(q-1)(q-2)}+\frac{6b^2}{(1+b)(q-2)(q-3)}-\left[\frac{b^2}{(1+b)(q-1)}+\frac{2b}{(1+b)(q-2)}\right]^2$$

where $q>3$

3) Coefficient of Skewness

$$Skewness(X)\ =\ \frac{E(X^3)-3E(X)Var(X)-\left(E(X)\right)^3}{\left(Var(X)\right)^{\frac{3}{2}}}$$

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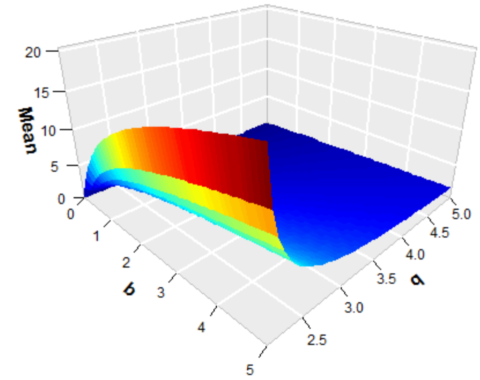
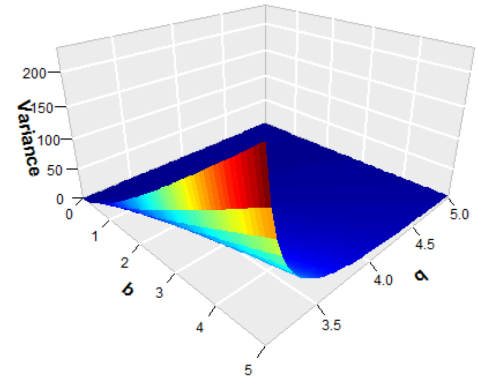
4) Coefficient of Kurtosis

$$Kurtosis(X) = \left[E(X^4) - 4E(X^3)E(X) + 6E(X^2)(E(X))^2 - 3(E(X))^4\right] / (Var(X))^2$$

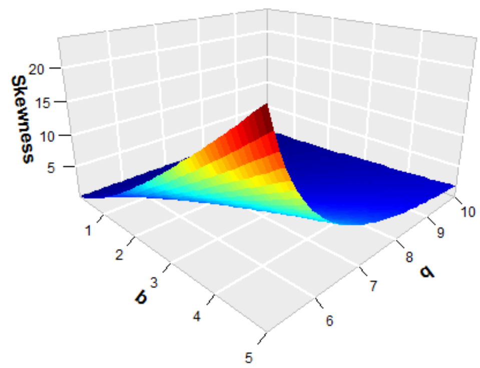
The mean, variance, coefficient of skewness and coefficient of kurtosis for new mixture Lomax distribution for different values of the parameters b and q in Figure 5.



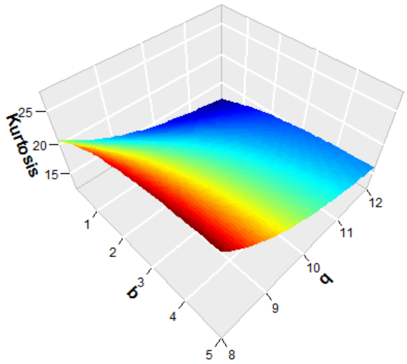
(A) Mean



(B) Variance



(C) Coefficient of Skewness



(D) Coefficient of Kurtosis

Figure 5. Plot of the mean (A), variance (B), coefficient of skewness (C) and coefficient of kurtosis (D) for new mixture Lomax distribution : versus b and q.

3. Parameters estimation

In this section, we consider maximum likelihood estimation to estimate the involved parameters of new mixture Lomax distribution. The likelihood function of new mixture Lomax distribution with b and q parameter is defined in Definition 2.

Definition 2 . Suppose that a random sample $X_1, X_2, ..., X_n$ is collected from new mixture Lomax distribution, Then the likelihood function of the observed sample is given by

$$L(b,q;x_1,x_2,x_3,...,x_n) = \prod_{i=1}^{n}\left[\frac{q(b^2 + qx_i - x_i)}{(b^3+b^2)(1+x_i/b)^{1+q}}\right].$$

The maximum likelihood estimates of the parameters are obtained by direct maximization of the log-likelihood function is produced as follows;

1) The likelihood function and log-like likelihood of new mixture Lomax distribution are, respectively, given by

$$L(b,q) = L(b,q;x_1,x_2,x_3,...,x_n) = \prod_{i=1}^{n}\left[\frac{q(b^2 + qx_i - x_i)}{(b^3+b^2)(1+x_i/b)^{1+q}}\right]$$

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$$\log L(b,q) = n\log q + \sum_{i=1}^{n} \log(b^2 + qx_i - x_i) - n\log\left(b^3 + b^2\right) - (1+q)\sum_{i=1}^{n} \log\left(1 + x_i / b\right).$$

2) Finding the optimal values of the parameters is obtained by differentiating in $\log L(b,q)$ with b and q. Then, it gives rise to following equations:

$$\frac{d}{db}\log L(b,q) = \sum_{i=1}^{n} \frac{2b}{(b^2 + qx_i - x_i)} - \frac{n(3b^2 + 2b)}{\left(b^3 + b^2\right)} + (1+q)\sum_{i=1}^{n} \frac{b^{-2}}{\left(1 + x_i / b\right)} \quad (1)$$

$$\frac{d}{dq}\log L(b,q) = \frac{n}{q} + \sum_{i=1}^{n} \frac{x_i}{(b^2 + qx_i - x_i)} - \sum_{i=1}^{n} \log\left(1 + x_i / b\right) \quad (2)$$

3) The MLE solutions of $\hat{b},\hat{q}$ can be obtained by equating the (1) and (2) to zero and solving the resulting equations simultaneously using a numerical procedure with the Newton-Raphson method.

4. Numerical Result

In this section, we have considered a dataset corresponding to remission times (in months) of a random sample of 128 bladder cancer patients given in Lee and Wang (2003). We have fitted the new mixture Lomax distribution to the dataset using MLE, and compared new mixture Lomax distribution with Lomax, old mixture Lomax and length biased Lomax distributions which are showed in Table 1. The model selection is carried out using AIC (Akaike information criterion).

Table 1. Maximum likelihood Estimate and AIC

Distribution	**Estimate parameters**			**AIC**
	$\hat{p}$	$\hat{b}$	$\hat{q}$	
Lomax	-	11.09	94.47	799.13
Length biased Lomax	-	4.33	10.97	793.84
Old mixture Lomax	0.21	15.75	75.11	785.34
New mixture Lomax	-	12.35	45.12	772.21

We fitted the Lomax, length biased Lomax, old mixture Lomax and new mixture Lomax distributions to this data set. The maximum likelihood estimation was used. We obtained the estimates parameters and AIC statistic for all distributions are shown in Table 1. The AIC statistic is shown that the new mixture Lomax distributions distribution is the best fit for this data.

5. Conclusion

This paper provides a new mixture Lomax distribution for lifetime data. Various interesting mathematical statistics properties of new mixture Lomax distribution such as its hazard rate function survival function, moment about origin and expressions for mean, variance, coefficient of skewness and coefficient of kurtosis which have been discussed. We estimate parameters of new mixture Lomax distribution by using maximum likelihood estimation method. In application, we compare the fit of the new mixture Lomax distribution with Lomax, length biased Lomax and old mixture Lomax distributions by real data. The AIC statistics indicates that the new mixture Lomax distribution is best fit for real data.

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